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SOME CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION FUNCTION

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Abstract. Let X be a nonnegative random variable and let $[x]$ denote the integer part of x. The main result of the paper is the following characterization: *X is exponentially distributed iff $[\alpha X]$ and $\alpha X - [\alpha X]$ are mutually independent for every $\alpha > 0$.* Some modifications of this theorem are also considered.

1. Results. Let X be a nonnegative random variable, and let $F(x) = \Pr(X < x)$ be its probability distribution function. Assume that the distribution is not concentrated at one atom. We say that X is *exponentially distributed* if $F(x) = 1 - e^{-\lambda x}$ $(x > 0)$ for some $\lambda > 0$. We say that X is *geometrically distributed* if $\Pr(X = k) = pq^k$, $k = 0, 1, \ldots$, for some $0 < p < 1$, $q = 1 - p$. Denote by $[x]$ the integer part of x.

The main result of the paper is the following characterization of the exponential probability distribution function:

THEOREM 1. *X is exponentially distributed iff, for every $\alpha > 0$, $[\alpha X]$ and $\alpha X - [\alpha X]$ are mutually independent.*

The random variables $[\alpha X]$ and $\alpha X - [\alpha X]$, separately considered, may be used to the characterization of the exponential probability distribution function.

THEOREM 2 (Bosch [1]). *X is exponentially distributed iff, for every $\alpha > 0$, $[\alpha X]$ is geometrically distributed.*

THEOREM 3. *X is exponentially distributed iff, for every $\alpha > 0$, $\alpha X - [\alpha X]$ has the truncated exponential probability distribution function.*

The modified version of Bosch's theorem is given by Riedl [3]. Theorem 1 has its discrete version and its continuous version formulated in terms of the renewal theory.

THEOREM 4. *Let X be a nonnegative integer-valued random variable. X is* *geometrically distributed iff, for every* $a = 1, 2, \ldots,$ $[X/a]$ *and* $X - a[X/a]$ *are mutually independent.*

THEOREM 5. *Let* $X, Y_1, Y_2, \ldots$ *be independent nonnegative random variables, let* X *have an absolutely continuous probability distribution function with bounded and continuous density, and let* $Y_1, Y_2, \ldots,$ *have a common probability distribution function with the finite expected value. Let* $N(t) = \max(n\colon Y_1 + \ldots + Y_n \leqslant t)$ *and* $R(t) = t - (Y_1 + \ldots + Y_{N(t)})$, $t \geqslant 0$, *be the renewal process and the residual life process, respectively.* X *is exponentially distributed iff, for every* $\alpha > 0$, $N(\alpha X)$ *and* $R(\alpha X)$ *are mutually independent.*

In the proofs which now follow, we limit our considerations merely to the "only if" part.

2. Proof of Theorem 1. Write $N = [\alpha X]$ and $R = \alpha X - N$. Let $\mathscr{B}$ be the σ-field of Borel sets on $[0, 1]$. Define $\alpha(B+\beta)$ for $\alpha > 0$, $-\infty < \beta < \infty$, in such a manner that $x \in B$ iff $\alpha(x+\beta) \in \alpha(B+\beta)$. We have

$$\Pr(N = n) = \Pr(n \leqslant \alpha X < n+1) = F\left(\frac{n+1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right),$$

$$\Pr(R \in B) = \Pr\left(X \in \bigcup_{n=0}^{\infty} \frac{B+n}{\alpha}\right) = \sum_{n=0}^{\infty} \Pr\left(X \in \frac{B+n}{\alpha}\right),$$

$$\Pr(N = n,\ R \in B) = \Pr\left(X \in \frac{B+n}{\alpha}\right), \quad n = 0, 1, \ldots,\ B \in \mathscr{B},\ \alpha > 0.$$

The independence condition for N and R may be written as

$$(1) \quad \Pr\left(X \in \frac{B+n}{\alpha}\right) = \left(F\left(\frac{n+1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right) \sum_{k=0}^{\infty} \Pr\left(X \in \frac{B+k}{\alpha}\right),$$
$$n = 0, 1, \ldots,\ B \in \mathscr{B},\ \alpha > 0.$$

If $B = [0, y)$, $0 \leqslant y \leqslant 1$, then (1) has the form

$$(2) \quad F\left(\frac{n+y}{\alpha}\right) - F\left(\frac{n}{\alpha}\right) = \left(F\left(\frac{n+1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right) \sum_{k=0}^{\infty} \left(F\left(\frac{k+y}{\alpha}\right) - F\left(\frac{k}{\alpha}\right)\right),$$
$$n = 0, 1, \ldots,\ 0 \leqslant y \leqslant 1,\ \alpha > 0.$$

For $n = 0$ we have

$$(3) \quad F\left(\frac{y}{\alpha}\right) = F\left(\frac{1}{\alpha}\right) \sum_{k=0}^{\infty} \left(F\left(\frac{k+y}{\alpha}\right) - F\left(\frac{k}{\alpha}\right)\right), \quad 0 \leqslant y \leqslant 1,\ \alpha > 0.$$

For α such that $F(1/\alpha) > 0$ we have

$$(4) \quad F\left(\frac{n+y}{\alpha}\right) - F\left(\frac{n}{\alpha}\right) = \left(F\left(\frac{n+1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right) F\left(\frac{y}{\alpha}\right) / F\left(\frac{1}{\alpha}\right),$$
$$n = 1, 2, \ldots,\ 0 \leqslant y \leqslant 1,\ \alpha > 0.$$

Let $F = pF_d + qF_s + rF_a$, where F_d, F_s and F_a are discrete, singular and absolutely continuous components, $p \geqslant 0$, $q \geqslant 0$ and $r \geqslant 0$, $p+q+r = 1$, are the weights of one.

Let $B = \{x_i, i = 1, 2, \ldots\}$ be the support of the discrete component of the distribution function F, $\Pr(X \in B) = p$. Consider $\alpha > 0$ such that $0 < F(1/\alpha) < 1$. For nondegenerate F the set α which satisfies that condition contains some interval. Define $B_\alpha = \{\alpha x_i - [\alpha x_i], i = 1, 2, \ldots\}$. We have

$$\Pr\left(X \in \frac{1}{\alpha}(B_\alpha + n)\right) = \Pr\left(X = x_i\colon \frac{n}{\alpha} \leqslant x_i < \frac{n+1}{\alpha}\right)$$

$$= pF_d\left(\frac{n+1}{\alpha}\right) - pF_d\left(\frac{n}{\alpha}\right), \qquad n = 0, 1, \ldots$$

From (1) for $n = 0$ and $B = B_\alpha$ it follows that

$$pF_d\left(\frac{1}{\alpha}\right) = \left(pF_d\left(\frac{1}{\alpha}\right) + qF_s\left(\frac{1}{\alpha}\right) + rF_a\left(\frac{1}{\alpha}\right)\right)p,$$

which implies $p = 0$ or $p = 1$.

Let $p = 0$ and B be the support of the singular component of the distribution F (e.g. B is a set of the Lebesgue measure zero), $B \subset [0, \infty)$, and $\Pr(X \in B) = q$. Let

$$B_\alpha = \bigcup_{k=0}^{\infty} \alpha\left(B \cap \left[\frac{k}{\alpha}, \frac{k+1}{\alpha}\right) - \frac{k}{\alpha}\right).$$

We have

$$\Pr\left(X \in \frac{1}{\alpha}(B_\alpha + n)\right) = \Pr\left(X \in B_\alpha \cap \left[\frac{n}{\alpha}, \frac{n+1}{\alpha}\right)\right)$$

$$= qF_s\left(\frac{n+1}{\alpha}\right) - qF_s\left(\frac{n}{\alpha}\right), \qquad n = 0, 1, \ldots$$

From (1) for $n = 0$ and $B = B_\alpha$ it follows that

$$qF_s\left(\frac{1}{\alpha}\right) = \left(qF_s\left(\frac{1}{\alpha}\right) + rF_a\left(\frac{1}{\alpha}\right)\right)q,$$

which implies $q = 0$ or $q = 1$.

Now we prove that $0 < F(x) < 1$ for $x > 0$. The conditions $F(a) = 0$, $F(a+0) > 0$ for some $a > 0$ and (4) imply that F is discrete and generated by $\Pr(X = ka) = p_k \geqslant 0$, $k = 1, 2, \ldots$, $p_1 + p_2 + \ldots = 1$. Putting $1/\alpha > a$ and such that a/α is irrational, from (1) for $n = 0$ and $B = \{\alpha a\}$ it follows that $\Pr(X = k/\alpha+a,\ k = 1, 2, \ldots) > 0$, which does not hold. The conditions $F(a) < 1$, $F(a+0) = 1$ and (3) imply

$$F\left(\frac{a}{2}-\varepsilon\right) = F\left(\frac{a}{2}+2\varepsilon\right)\left(F\left(\frac{a}{2}-\varepsilon\right)+1-F\left(\frac{a}{2}+2\varepsilon\right)\right), \quad \text{where } 0 < \varepsilon < \frac{a}{4}.$$

Hence $F(\frac{1}{2}a-\varepsilon) = 1$ or $F(\frac{1}{2}a+2\varepsilon) = 0$, which does not hold.

It is obvious that the derivative of the probability distribution function exists almost surely. From (4) it follows that $f^+(0) = \lim_{t\downarrow 0} F(t)/t$ exists.

Now we prove that $f^+(0) > 0$.

Write $1-F = \bar{F}$. From (3) it follows that

$$F\left(\frac{y}{\alpha}\right) \geqslant F\left(\frac{1}{\alpha}\right)\left(F\left(\frac{y}{\alpha}\right)+F\left(\frac{1+y}{\alpha}\right)-F\left(\frac{1}{\alpha}\right)\right),$$

that is

$$F\left(\frac{y}{\alpha}\right) \geqslant \left(F\left(\frac{1+y}{\alpha}\right)-F\left(\frac{1}{\alpha}\right)\right)F\left(\frac{1}{\alpha}\right)/\bar{F}\left(\frac{1}{\alpha}\right), \quad 0 \leqslant y \leqslant 1,\ \alpha > 0.$$

For fixed $x > 0$ substitute $y = \alpha x$, $1/\alpha = a$. If $0 \leqslant y \leqslant 1$, then $0 \leqslant x \leqslant 1/\alpha$, and we have

$$F(x) \geqslant \big(F(a+x)-F(a)\big)F(a)/\bar{F}(a), \quad 0 \leqslant x \leqslant a,$$

which implies

$$\begin{aligned} F(x) &\geqslant \sup_{a\geqslant x}\big(F(a+x)-F(a)\big)F(a)/\bar{F}(a) \\ &= \sup_{B-x>A\geqslant x}\ \sup_{A\leqslant a<B-x}\big(F(a+x)-F(a)\big)F(a)/\bar{F}(a) \\ &\geqslant \sup_{B-x>A\geqslant x}\big(F(A)/\bar{F}(A)\big)\sup_{A\leqslant a<B-x}\big(F(a+x)-F(a)\big) \\ &\geqslant \sup_{B-x>A\geqslant x}\frac{F(A)}{\bar{F}(A)}\frac{F(B)-F(A)}{B-A}x, \end{aligned}$$

and, finally,

$$\frac{F(x)}{x} \geqslant \sup_{B-x>A\geqslant x}\frac{F(A)}{\bar{F}(A)}\frac{F(B)-F(A)}{B-A} > 0.$$

From (4) it follows that if n/α is the point of existence of the derivative of F, then

$$f\left(\frac{n}{\alpha}\right) = \left(F\left(\frac{n+1}{\alpha}\right)-F\left(\frac{n}{\alpha}\right)\right)f^+(0)/F\left(\frac{1}{\alpha}\right).$$

Hence the absolute continuous component of F has the positive weight. There remains the case $p=0$, $q=0$, $r=1$ (e.g. $F=F_a$).

From (4) we get

$$f\left(\frac{n+y}{\alpha}\right)=\left(F\left(\frac{n+1}{\alpha}\right)-F\left(\frac{n}{\alpha}\right)\right)f\left(\frac{y}{\alpha}\right)/F\left(\frac{1}{\alpha}\right)$$

which implies

$$(5) \qquad f\left(\frac{n+y}{\alpha}\right)f^{+}(0)=f\left(\frac{n}{\alpha}\right)f\left(\frac{y}{\alpha}\right), \qquad n=1,2,\ldots;\ 0<y<1,\ \alpha>0.$$

It is obvious that the unique solution of equation (5) in the class of integrable functions is $f(x)=\lambda e^{-\lambda x}$ $(x>0)$ for some $\lambda>0$.

3. Proof of Theorem 3. Let X have the probability distribution function F. Then, for $R=\alpha X-[\alpha X]$, we have

$$H(y)=\Pr(R<y)=\sum_{n=0}^{\infty}\left(F\left(\frac{n+y}{\alpha}\right)-F\left(\frac{n}{\alpha}\right)\right), \qquad 0\leqslant y\leqslant 1,\ \alpha>0.$$

We have assumed that H is a truncated exponential probability distribution function, e.g.

$$H(y)=(1-e^{-\lambda(\alpha)y})/(1-e^{-\lambda(\alpha)}), \qquad 0\leqslant y\leqslant 1,$$

where $\lambda(\alpha)$ is a parameter which depends on α only.

Taking the limit $H(y)$ if $\alpha\to 0$ and $y/\alpha\to x$, since $F(y/\alpha)\leqslant H(y)\leqslant F(y/\alpha)+1-F(1/\alpha)$, we get

$$F(x)=\lim_{\alpha\to 0}\frac{1-e^{-\lambda(\alpha)\alpha x}}{1-e^{-\lambda(\alpha)}}.$$

Hence the limits, for $\alpha\to 0$, $\lim\lambda(\alpha)\alpha=\lambda$ and $\lim\lambda(\alpha)=\infty$ exist. Finally, we have $F(x)=1-e^{-\lambda x}(x\geqslant 0)$, where $\lambda>0$ for the nondegenerate case.

4. Proof of Theorem 4. Let $\Pr(X=k)=p_k$, $k=0,1,\ldots$ Then, for $a=1,2,\ldots$, we have $\Pr([X/a]=n,\ X-a[X/a]=i)=\Pr(X=an+i)=p_{an+i}$, $n=0,1,\ldots;\ i=0,1,\ldots,a-1$. The independence condition for $[X/a]$ and $X-a[X/a]$ is equivalent to

$$p_{an+i}=\Big(\sum_{j=0}^{a-1}p_{an+j}\Big)\Big(\sum_{k=0}^{\infty}p_{ak+i}\Big), \qquad n=0,1,\ldots;\ i=0,1,\ldots,a-1;\ a=1,2,\ldots,$$

whence

$$(6) \qquad p_{am+i+1}=p_{am+i}\,q_i, \qquad i=0,1,\ldots,a-2;\ m=0,1,\ldots;\ a=1,2,\ldots,$$

where q_i does not depend on m.

In particular, for $a = 2$, we have

$$p_{2n+1} = p_{2n} q, \quad n = 0, 1, \ldots, \tag{7}$$

where q does not depend on n.

Let $a = 4k+1$. If $n = 0, 1, \ldots, 2k-1$ in (7), and $m = 0$, $i = 0, 2, \ldots, 4k-2$ in (6), then $q_i = q$ for $i = 0, 2, \ldots, 4k-2$. If $n = 2k+1, 2k+2, \ldots, 4k$ in (7), and $m = 1$, $i = 1, 3, \ldots, 4k-1$ in (6), then $q_i = q$ for $i = 1, 3, \ldots, 4k-1$. We have $p_{i+1} = p_i q$ for $i = 0, 1, \ldots, 4k-1$. Since k is arbitrary, we have $p_i = pq^i$, $i = 0, 1, \ldots$

5. Proof of Theorem 5. Let $G_0(x) = 1_{(0,\infty)}(x)$, $G(x) = \Pr(Y_1 < x)$, $G_n(x) = \Pr(Y_1 + Y_2 + \ldots + Y_n < x)$, $x > 0$, $n = 1, 2, \ldots$, $\bar{G} = 1 - G$, $\mathrm{E}Y_1 = \mu_1$. Assuming the existence of the probability density function f, we improve the joint density of $N(\alpha X)$ and $R(\alpha X)$:

$$\frac{d}{dy}\Pr\big(N(\alpha X) = n,\ R(\alpha X) < y\big) = \int_0^\infty \frac{1}{\alpha} f\left(\frac{u+y}{\alpha}\right)\bar{G}(y)\, dG_n(u),$$

$$\Pr\big(N(\alpha X) = n\big) = \int_0^\infty \int_0^\infty \frac{1}{\alpha} f\left(\frac{u+y}{\alpha}\right)\bar{G}(y)\, dy\, dG_n(u)$$

$$= \int_0^\infty \frac{1}{\alpha} f\left(\frac{z}{\alpha}\right)\big(G_n(z) - G_{n+1}(z)\big)\, dz,$$

$$\frac{d}{dy}\Pr\big(R(\alpha X) < y\big) = \int_0^\infty \frac{1}{\alpha} f\left(\frac{u+y}{\alpha}\right)\bar{G}(y) \sum_{k=0}^\infty dG_k(u)$$

$$= \int_0^\infty \frac{1}{\alpha} f\left(\frac{u+y}{\alpha}\right)\bar{G}(y)\, dH_G(u), \qquad n = 0, 1, \ldots,\ y \geqslant 0,\ \alpha > 0,$$

where

$$H_G(u) = \sum_{k=0}^\infty G_k(u) = \mathrm{E}N(u), \qquad u \geqslant 0.$$

The independence condition of $N(\alpha X)$ and $R(\alpha X)$ has the form

$$\int_0^\infty f\left(\frac{u+y}{\alpha}\right)\bar{G}(y)\, dG_n(u) = \left(\int_0^\infty f\left(\frac{y}{\alpha}\right)\big(G_n(y) - G_{n+1}(y)\big)\, dy\right) \int_0^\infty f\left(\frac{u+y}{\alpha}\right)\bar{G}(y)\, dH_G(u)$$

$$n = 0, 1, \ldots,\ y \geqslant 0,\ \alpha > 0.$$

For $n = 0$ we have

$$f\left(\frac{y}{\alpha}\right)\bar{G}(y) = \left(\int_0^\infty f\left(\frac{y}{\alpha}\right)\bar{G}(y)\, dy\right)\left(\int_0^\infty f\left(\frac{u+y}{\alpha}\right)\bar{G}(y)\, dH_G(u)\right).$$

Hence, for $\bar{G}(y) > 0$,

$$\frac{f(y/\alpha)}{\int_0^\infty f((u+y)/\alpha)\,dG_n(u)} = \frac{\int_0^\infty f(y/\alpha)\bar{G}(y)\,dy}{\int_0^\infty f(y/\alpha)\big(G_n(y)-G_{n+1}(y)\big)\,dy}. \tag{8}$$

Putting $y = 0$ in (8), we get a further simplification:

$$\frac{f(y/\alpha)}{\int_0^\infty f((u+y)/\alpha)\,dG_n(u)} = \frac{f(0)}{\int_0^\infty f(u/\alpha)\,dG_n(u)}, \tag{9}$$

$$n = 1, 2, \ldots, y \geqslant 0, \alpha \geqslant 0, \bar{G}(y) > 0.$$

Let $x > 0$, $m = [n\mu_1/x]$. We have

$$G_n(mu) = \Pr(Y_1 + \ldots + Y_n < mu) = \Pr\left(\frac{1}{n}(Y_1 + \ldots + Y_n) < \frac{mu}{n}\right).$$

Since $m/n \to \mu_1/x$, we have $G_n(mu) \to 1_{(x,\infty)}(u)$. For bounded and continuous f (see [2], p. 254) we have

$$\frac{f(y/\alpha)}{f((y+x)/\alpha)} = \frac{f(0)}{f(x/\alpha)}, \quad y \geqslant 0, x > 0, \alpha > 0. \tag{10}$$

Substituting $u := mu$, $\alpha := m\alpha$, $y := my$ in (9) and taking the limit if $n \to \infty$, we get

$$\frac{f(y/\alpha)}{f((y+x)/\alpha)} = \frac{f(0)}{f(x/\alpha)}, \quad y \geqslant 0, x > 0, \alpha > 0. \tag{10}$$

The unique continuous solution of (10) is $f(x) = \lambda e^{-\lambda x}$ $(x > 0)$ for some $\lambda > 0$.

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